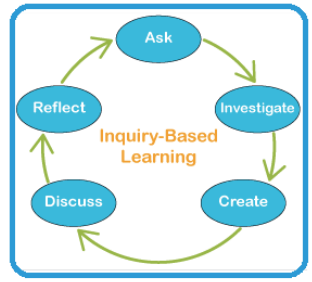
Quantitative Literacy - Problems that Motivate

Christopher Shaw, Columbia College Chicago

Abstract

In this article, I present a selection of problem types, with examples, that have been used with some success to motivate the topics in a quantitative literacy class so that learners may begin doing mathematics without period of discussion beforehand. In their original use, the setting for these problems is a small class at a large liberal arts college, but I hope that the work described within will still have some value for teachers of high school students, especially in classes where many learners struggle to appreciate mathematics.



The approach outlined in this article is inspired by the methods of Inquiry-Based Learning, "a teaching approach which intends to promote learning by engaging students in any of the processes or activities typically involved in scientific research…"(Ariza, et al., 2012). Scientific research itself has been documented since the dawn of history, but the term Inquiry-Based Learning (IBL), as well as the application of this philosophy to K-16 education, is comparatively recent. In its purest form, IBL may take the form of a classroom environment in which the instructor spends no time on traditional lecture: students ask research questions, and acquire topical and conceptual knowledge along the way to solving the problems on their own. They then present their work and actively listen to presentations from their peers. In a more diluted form, IBL tools can be integrated into a lecture model to support alternative development of concepts. Proponents of IBL argue that students benefit from active involvement over the traditional lecture model, particularly in the mathematics classrooms in Liberal Arts environments (Fleron, 2013). Also, although developed independently, the philosophy behind IBL aligns with the Common Core Standards for Mathematical Practice, particularly the Mathematical Practice Standard 1: "Make sense of problems and persevere in solving them" (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). A great deal of resource material on IBL, including lecture notes and research on outcomes assessment, is hosted online by the research collectives *Discovering the Art of Mathematics* (https://www.artofmathematics.org) and *The Center for Inquiry-Based Learning* at the University of Michigan (http://dept.math.lsa.umich.edu/ibl/index.html).

By way of introduction, I highlight several important factors that contribute to the particular character of this course. Our Quantitative Literacy course, designed for a college with a focus on visual and performing arts, includes units on set algebra and Venn diagrams, deductive reasoning, introductory algebra, general problem-solving, and topics in probability. Students often enter this course with a negative view of mathematics; in many cases, we encounter physical symptoms of mathematical anxiety. We also have many students who, despite being prepared to enter a more advanced college mathematics track, may choose to take this course in order to ease their overall intellectual burden. These conditions contribute to the great diversity in our student's mathematical ability and preparation. A further factor is that our mathematics courses are often taught by part-time instructors who must split their time among several institutions. Thus, even though our classes tend to be on the smaller side, curricula are designed to facilitate streamlined adoption by instructors. To achieve uniformity across sections, this course still uses a formal textbook and has some component of lecture. However, we often introduce new topics through the lens of IBL, beginning on the first day of class, solving problems in groups, with no lecture, and a minimal assumption of prior mathematical knowledge.

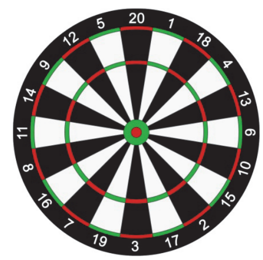
We introduce new topics as much as possible through the lens of IBL, beginning on the first day of class, solving problems in groups, with no lecture, and a minimal assumption of prior mathematical knowledge.

Problems that Motivate

Below, I discuss four examples of problem types that our sections use on the first day of class, along with some typical student solutions and discussion of the value of these problems to motivate course material. A key to the approach is that we present the students with these problems immediately after we discuss the course syllabus, meaning that there is no mathematical discussion in class prior to the students beginning to do mathematics on their own.

Problem Type 1: Expected value

> Suppose that you are a dart player of medium skill. When you aim a dart at a particular number, you hit that value about half of the time; the other half of your throws hit one of the values immediately adjacent. (For example, when aiming at the 14, about half of your throws will land on 14, a quarter will land on 11, and a quarter will land on 9.)
> Ignoring the bulls-eye, what number should you target in order to achieve a consistently high score over many throws?



Discussion: This problem, without using the terms "probability" or "expected value," invites students to calculate the expected value of a dart throw given a set of probabilities. In my classroom, usually only a few students are intuitively able to begin working on the problem without any prompting. After a few minutes, I usually add the following hint to the problem: "Replace the phrase "many throws" with 100 throws." Each of the solution samples below (recreated to be more compact and legible) came from students who achieved the correct answer: aiming at the 7 is likely to give the highest score over the course of throwing. However, each answer illustrates a different way of thinking about the problem.

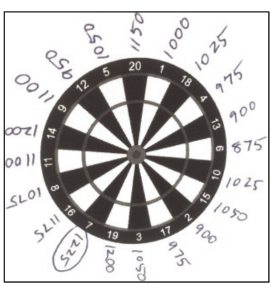


Figure 1A.

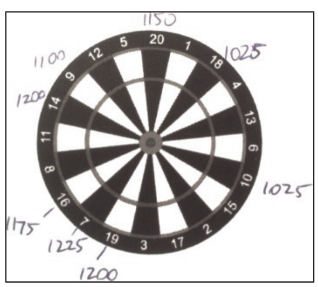


Figure 1B.

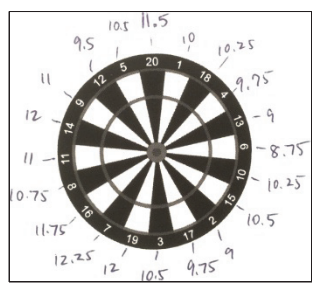


Figure 1C.

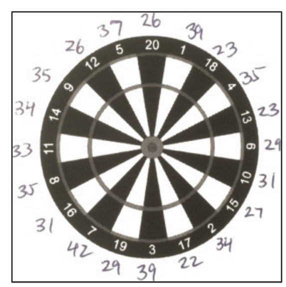


Figure 1D.

- Figure 1A shows the most exhaustive calculation: the student who produced this diagram calculated the expected final score from aiming at each region, and determined that the 7 yielded the highest score.
- In Figure 1B, the student first streamlined the process by throwing out some regions that she thought would obviously *not* end up with the highest total. Otherwise, the student did the same calculations as the student who produced Figure 1A.
- The student who created Figure 1C computed the *expected value* of a single dart throw for each region, rather than calculating the final score for 100 darts thrown. One feature of this problem is that, even if a solver is familiar with expected value, the student is not able to get to the answer without doing the same work as a student who is not familiar with expected value. In other words, knowing the concept does not provide the student with a shortcut or trick to solving the problem.

- Finally, Figure 1D was produced by a student who got lucky: this student summed each target value with its two adjacent values, without weighting the values by their probabilities. In this case it turned out that the highest value recorded in the figure coincided with the correct answer, but that would not be true for every possible probability given, as we discussed once the students had presented their work.

In our discussion of this problem, the students are usually interested in finding out what would happen if they had better or worse aim. This is a great opportunity to develop a lesson using spreadsheets for calculation, but since such a unit is not included in my course, I often send them a link to a casual academic paper (Tibshirani, et al., 2011) about dart strategies for various skill levels.

Problem Type 2: Systems of linear equations.

MEDIA | 'The Interview' Brings In $15 Million on Web

MEDIA

'The Interview' Brings In $15 Million on Web

By MICHAEL CIEPLY DEC. 28, 2014

Email

Share

Tweet

Save

LOS ANGELES — "The Interview" generated roughly $15 million in online sales and rentals during its first four days of availability, Sony Pictures said on Sunday.

Sony did not say how much of that total represented $6 digital rentals versus $15 sales. The studio said there were about two million transactions over all.

"The Interview," a farce that depicts the killing of the North Korean

Read the first two paragraphs of the article included above. Can you determine how much of the $15 million came from rentals and how much from sales?

Figure 2. Algebra problem with image embedded (Cieply, 2014)

Discussion: This specific problem has not been classroom-tested yet, because at the time of this writing, the news article was current. However, I chose to include it because it is similar to problems I have used in the past, and it has been the subject of some analysis in popular social media. In particular, Dan Meyer (Meyer, 2014) compared the story above to a previous editorial printed by the New York Times, in which the author questioned whether Algebra was a necessary part of the mathematics curriculum (Hacker, 2012). This problem is powerful as it allows for the opportunity to provide entry points to mathematical topics, such as:

- Solving problems through educated guess-and-check methods (this is how I recommend my students to proceed when I guide them in class);
- Algebraic solutions to systems of linear equations: this method works for the problem, and if students are familiar with algebraic solving, it provides a good opportunity to implement the technique to solve a real-life problem;

- Rounding and approximation: in my experience, working with similar problems in class, students know that “roughly $15 million” can refer to a wide band of possible values, and that using an estimate for the total amount means that a solution to the algebraic problem at hand will carry an element of inherited uncertainty.

Using past experience with similar problems as a guide, I present a typical expected student solution to this problem, using educated guess-and-check, as shown in Figure 3:

	Rentals (millions)	**Sales (millions)**	**Revenue (millions)**	Assessment
Guess 1	1	1	$6 + $15 = $21	Too high, decrease sales, increase rentals.
Guess 2	1.5	0.5	$9 + $7.5 = $16.5	Too high, decrease sales, increase rentals.
Guess 3	1.7	0.3	$10.2 + $4.5 = $14.7	Close enough approximation.

Figure 3. Table showing a typical guess-and-check solution

Again, this problem can be solved algebraically using a system of equations; in fact, many of my students do attempt to solve the equations algebraically. However, the goal of this task is not to teach students a symbolic algorithm; the goal is to imbue students with the more general idea that, given two different conditions on two unknown quantities, there may be enough information to determine the quantities. Further, this method of solution emphasizes that, even if they do not remember how to solve simultaneous equations algebraically, they still do have the basic logical tools needed to derive a solution.

Problem Type 3: Subsets

You have one drop each of four different colors of very powerful dyes. You plan to use them to make one paint color, by any number of the dyes to mix into a dish of pure white paint. How many distinct paint colors are possible?

Discussion: This problem is included because it helps to motivate counting with sets and Pascal’s triangle, both of which we discuss later in our semester. Importantly, however, the words set and subset are absent from the description of the problem – the key to accessing the solution is the understanding that every possible combination of the different dyes – even using no dye at all – will result in a different color. Two representative work samples appear in Figures 4A and 4B.

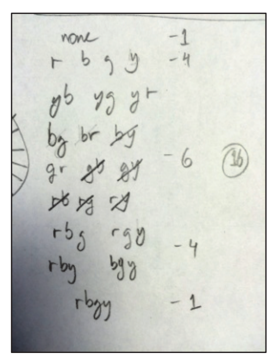


Figure 4A.

The solution method in Figure 4A, to the right, correctly counts up all 16 of the subsets of a set with cardinality 4 and, in so doing, avoided a common error: the most common mistake that I see when discussing this problem with students, or working with subsets in general, is the tendency to write a single subset down multiple times with the elements listed in a different order.

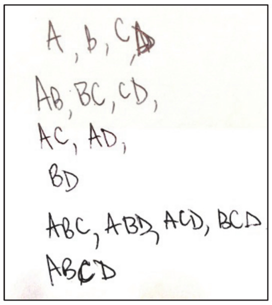


Figure 4B.

The solution in Figure 4B, to the left, illustrates another common error using the subset definitions, both in this problem and in problems that appear later- ignoring the empty set. (In the given problem, the empty set corresponds to using no dyes at all – yielding an unchanged bowl of white paint.)

Problem Type 4: Operational efficiency

Modern washer and dryer cycles equipped with sensors have different durations depending on the size of the load. Suppose you have one washer and one dryer, in which you have to launder a small load and a large load.

	Washer	**Dryer**
Small load	20 minutes	25 minutes
Large load	30 minutes	40 minutes

Does the order in which you do your laundry loads make a difference for the amount of time it takes to complete?

Discussion: This problem stands out from the others presented here because it calls upon pure reasoning to solve it. The dialogue below is representative of a common series of prompts that I use when presenting the problem. In this case, the student had a comparatively strong background in algebra.

Student:	It's the same either way!
Teacher:	That was fast. Why do you think so?
Student:	Because the times add up the same in any order. Either way, it takes 115 minutes: 20 + 25 + 30 + 40.
Teacher:	Say your first load went through the washer and you just put it into the dryer. What are you doing with the second load?

At this point in the conversation, the student realized that the washer and dryer can run at the same time, which is the key to solving the problem. He then worked out an answer by recalculating the total time either way. The axiom that governs the solution is that a load which has come out of the wash cannot go into the dryer until the dryer has finished running the cycle with the previous load. Once he realized that the problem could be worked out by role-playing the wash cycles, he got a correct answer. The best solution that I have received for this problem was the set of tables shown in Figures 5A and 5B.

	Washer	Dryer	Total time
Load 1	(Large) 30 min	---	30 min
Load 2	(Small) 20 min	(Large) 40 min	70 min
Load 3	---	(Small) 25 min	95 min

Figure 5A. Large, Small

	Washer	Dryer	Total time
Load 1	(Small) 20 min	---	20 min
Load 2	(Large) 30 min	(Small) 25 min	50 min
Load 3	---	(Large) 40 min	90 min

Figure 5B. Small, Large

Each Figure shows the total time elapsed from beginning the laundry until the last load can be removed from the dryer; Figure 5A calculates this time if the *large* load is done first and Figure 5B if the *small* load is done first. From these tables, it is clear that if one does the small load first, the total time to do the laundry is five minutes less. In my experience, the benefit to using an exercise like this is that it forces students to get past their inertia and just start working; there is no formula to call on, and the problem doesn't rely on any formal concept that a student has to memorize.

[T]he benefit to using an exercise like this is that it forces students to get past their inertia and just start working – there is no formula to call on, and the problem doesn't rely on any formal concept that a student has to memorize.

Conclusion

The activity described in this manuscript is meant to be used at the beginning of the first day of class, but in my experience, the memory of doing these problems can have a strong impact on the way students understand the mathematical concepts later on. For example, when I cover probability, usually during the last two weeks of the semester, my students remember the dartboard example and can visually make connections between their experience at the beginning of the semester and the expected value calculations that they do using more abstract spinners. The paint example recurs in a later chapter on sets, where I use similar examples to demonstrate the usefulness of abstract counting techniques.

These motivating problems are also fairly typical of the way in which I tend to begin any new unit, with effective results. As I continue to develop the curriculum for my quantitative literacy course, I continue to revise these activities. In the future, I hope to find more ways to get students actively working in class with as little time as possible spent in a traditional lecture model. As a first step toward implementing IBL in my curriculum, assigning a problem that covertly introduces a definition or technique is an effective way to motivate formalizing the concepts later in class.

References

Ariza, M., Quesada, A., Abril, A., & García, F. (2012, March). Inquiry based learning: why buying a car with a tree included? Enhancing science and mathematics learning. In *INTED2012 Proceedings, 6th International Technology, Education and Development Conference,* 601 – 609.

Cieply, M. (2014, December 28). The Interview Brings in $15 Million on Web. *New York Times.* Retrieved from http://www.nytimes.com/2014/12/29/business/media/the-interview-comes-to-itunes-store.html .

Fleron, J. (2013, June). The art of teaching mathematics to liberal arts students. *MAA Focus, 33*(3), 17 – 19.

Hacker, A. (2012, July 28). Is algebra necessary? *New York Times.* Retrieved from http://www.nytimes.com/2012/07/29/opinion/sunday/is-algebra-necessary.html .

Meyer, D. (Dec. 30, 2014, 10:27AM). "So glad the @nytimes found an answer to their question from 2012, "Is Algebra Necessary?" [h/t a commenter]". Tweet @ddmeyer.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics.* Washington, DC: Authors.
Tibshirani, R., Price, A., & Taylor, J. (2011, March). Don't try for the triple 20: Where to aim if you are bad at darts. *Significance Magazine, 8*(1), 46 – 48.

Christopher Shaw (cshaw@colum.edu) is Assistant Professor of Mathematics at Columbia College Chicago where he coordinates the Liberal Arts Mathematics curriculum. He is also co-editor of the *Illinois Mathematics Teacher Journal*. His mathematics research is in logic and his pedagogical focus is on entry-level college mathematics.